SOLUTIONS TO MID-TERM EXAM.

PROBLEM 1

Sol (a) See HW3Ex6.

$$\varphi : (x,y) \mapsto (u,v) = (xy, \frac{y}{x})$$

$$\varphi_* X = 2u\frac{\partial}{\partial u}, \quad \varphi_* Y = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}$$

(b) $$[X,Y] = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x}\right) - \left(y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

$$= y\frac{\partial}{\partial x} - y\frac{\partial}{\partial x} = 0.$$

So $\varphi_*[X,Y] = 0$.

$$[\varphi_* X, \varphi_* Y] = \left(2u\frac{\partial}{\partial u}\right)\left(uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}\right)$$
$$- \left(uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}\right)\left(2u\frac{\partial}{\partial u}\right)$$

$$= 2uv\frac{\partial}{\partial u} - 2uv\frac{\partial}{\partial v} = 0.$$

$$\Rightarrow [\varphi_* X, \varphi_* Y] = \varphi_*[X,Y].$$

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Rmk In general, $[\varphi_* X, \varphi_* Y] = \varphi_*[X,Y]$ always holds for $X, Y \in \Gamma(TM)$.

PROBLEM 2

Sol (a) $\frac{\partial F}{\partial x} = 2(y^2 + x(x-1)^2(x-2))\left[(x-1)^2(x-2) + 2x(x-1)(x-2)\right.$

$\left. + x(x-1)^2\right]$

$= 2(y^2 + x(x-1)^2(x-2))(x-1)\left[(x-1)(x-2) + 2x(x-2)\right.$

$\left. + x(x-1)\right]$

$= 2(y^2 + x(x-1)^2(x-2))(x-1)(4x^2 - 8x + 2)$

$\frac{\partial F}{\partial y} = 2(y^2 + x(x-1)^2(x-2)) \cdot 2y$

$= 4(y^2 + x(x-1)^2(x-2))y$

$\frac{\partial F}{\partial z} = 2z.$

$\Rightarrow$ For $F(\tilde{x}, \tilde{y}, \tilde{z}) = \varepsilon$, if $dF = 0$, then:

① $\frac{\partial F}{\partial z} = 2\tilde{z} \Rightarrow \tilde{z} = 0 \Rightarrow \tilde{y}^2 + \tilde{x}(\tilde{x}-1)^2(\tilde{x}-2) = \pm\sqrt{\varepsilon}$

② $\frac{\partial F}{\partial y} = 4(\tilde{y}^2 + \tilde{x}(\tilde{x}-1)^2(\tilde{x}-2))\tilde{y} \Rightarrow \tilde{y} = 0$

$\Rightarrow \tilde{x}(\tilde{x}-1)^2(\tilde{x}-2) = \pm\sqrt{\varepsilon}$

③ $\frac{\partial F}{\partial x} = 0 \Rightarrow (\tilde{x}-1)(4\tilde{x}^2 - 8\tilde{x} + 2) = 0$

$\tilde{x} = 1$, impossible; so $4\tilde{x}^2 - 8\tilde{x} + 2 = 0$

$\tilde{x} = 1 \pm \frac{\sqrt{2}}{2}$

So: let $f(x)=x(x-1)^2(x-2)$, take

$$0<\varepsilon<\min\left\{\left|f\left(1+\frac{\sqrt{2}}{2}\right)\right|^2, \left|f\left(1-\frac{\sqrt{2}}{2}\right)\right|^2\right\}$$

contradict with $\tilde{x}(\tilde{x}-1)^2(\tilde{x}-2)=\pm\sqrt{\varepsilon}$, so $dF\neq 0$

when $F(x,y,z)=\varepsilon$.

$\Rightarrow$ ε is a regular value of F

so $F^{-1}(\varepsilon)$ is an embedded submfd of $\mathbb{R}^3$

$\dim F^{-1}(\varepsilon)=3-rk(dF)=3-1=2$.

(b) $T_{(0,0,\sqrt{\varepsilon})}F^{-1}(\varepsilon)=\ker(dF_{(0,0,\sqrt{\varepsilon})})$.

$dF_{(0,0,\sqrt{\varepsilon})}=(0,0,2\sqrt{\varepsilon})$.

So $T_{(0,0,\sqrt{\varepsilon})}F^{-1}(\varepsilon)=\mathrm{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x}\Big|_{(0,0,\sqrt{\varepsilon})}, \frac{\partial}{\partial y}\Big|_{(0,0,\sqrt{\varepsilon})}\right\}$.

PROBLEM 3

Sol (a) Definition of LIE GROUPS:

A Lie group G is a smooth manifold and also a group, such that:

(i) $\mu: G\times G \to G, (g_1, g_2) \mapsto g_1 \cdot g_2$ is smooth;

(ii) $i: G \to G, g \mapsto g^{-1}$ is smooth.

(b) See Lec 1.

Lie group structure of S^3 is givin by unit quaternions

(c) Conclusion: $\nu_{S^3} \mathbb{R}^4 \simeq S^3 \times \mathbb{R}^1$ trival bundle on S^3.

Tangent bundle of S^3 is travial: $TS^3 \simeq S^3 \times \mathbb{R}^3$

($\Rightarrow$ Property of Lie groups)

$\mathbb{R}^4 \setminus \{0\}$ has a othogonal basis:

$N = (x^1, x^2, x^3, x^4) \rightsquigarrow$ (outward) normal vector of S^3

$$\left.\begin{aligned} e_1 &= (-x^2, x^1, -x^4, x^3) \\ e_2 &= (-x^4, x^3, -x^2, x^1) \\ e_3 &= (-x^3, x^4, x^1, -x^2) \end{aligned}\right\} \text{tangential to } S^3 \text{ on } S^3$$

So: $\nu_{S^3}\mathbb{R}^4 \simeq \underbrace{T\mathbb{R}^4|_{S^3}}_{\simeq S^3\times\mathbb{R}^4} \Big/ \underbrace{TS^3}_{\simeq S^3\times\mathbb{R}^3} \simeq S^3\times\mathbb{R}^1$

def of quotient bundle

(d) vector fields e_1, e_2, e_3 defined above are all vector fields on S^3 with no zero.

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PROBLEM 4

Sol (a) CRT: Let $f: M^m \to N^n$ be a constant rank map near p,

& $rk(f,p)=k$. Then exists cordinate charts

(φ, U) near p & (ψ, V) near $f(p)$ s.t.

$\psi \circ f \circ \varphi^{-1} : (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^k, 0, \cdots, 0)$.

Pf: any submersion is a open map.

Assume $f: M \to N$ is a submorsion.

$\forall U \overset{open}{\subseteq} M$, only need to show: $f(U) \overset{open}{\subseteq} N$.

$\Rightarrow$ For any $p \in U$, $\exists$ nbhd U_p of p, U_p is the cordinate chart w.r.t. p in CRT. WLOG $U_p \subset U$.

CRT $\Rightarrow f(U_p) \overset{open}{\subseteq} N$. Thus $f(U_p) \subset f(U)$, $f(U) \overset{open}{\subseteq} N$. #

(b) See Hw4 Ex3.

(c) GCRT If $f: M \to N$ has constant rank globally, then:

(i) f surjective $\Rightarrow$ submersion;

(ii) f injective $\Rightarrow$ immersion;

(iii) f bijective $\Rightarrow$ local diffeomorphism.

(d): First prove: φ is a constant map.

$\Rightarrow \forall g \in G$, set $\varphi(g) = h \in H$. Let $R_{g^{-1}}: G \to G$, $x \mapsto g^{-1}x$ &

$R_h: H \to H$, $y \mapsto hy$, then $R_{g^{-1}} \in \mathrm{Diff}(G)$, $R_h \in \mathrm{Diff}(H)$.

Near g, $\varphi := R_h \circ \varphi \circ R_{g^{-1}} \Rightarrow (d\varphi)_g = (dR_h)_{e_H} \circ (d\varphi)_{e_G} \circ (dR_{g^{-1}})_g$

where e_G, e_H are unit elements of G, H.

So $rk(d\varphi)_g = rk(d\varphi)_{e_G} = \text{constant}$, $\forall g \in G$.

As φ is a group isomorphism, then $rk(\varphi) = \dim(G) = \dim(H)$.

$\Rightarrow$ If NOT, by CRT, φ locally is not injective & surjective.

Then:

① $rk(\varphi) = \dim(G) = \dim(H) \Rightarrow \varphi$ submersion $\Rightarrow \varphi$ open map.

φ bijective + open map $\Rightarrow \varphi^{-1}$ exists & cts.

② $rk(\varphi) = \dim(G) = \dim(H) \Rightarrow d\varphi$ locally invertible.

<u>IFT</u> $\Rightarrow \varphi^{-1}$ is smooth.

So $\varphi: G \to H$ is a diffeomorphism.

(Or: by GCRT, bijective $\Rightarrow$ diffeomorphism) #

PROBLEM 5

Sol (a) CARTAN'S MAGIC FORMULA:

$$\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega, \quad X \in \Gamma(TM), \ \omega \in \Omega^k(M).$$

(b) Definition of HAMILTON VECTOR FIELD:

Hamilton vector field X_H generated by H is defined by

$$\omega(X_H, \cdot) := -dH(\cdot).$$

(c) First prove: $\frac{d}{dt}((\varphi_H^t)^* \omega)_p = (\varphi_H^t)^* (\mathcal{L}_{X_H} \omega)_{\varphi_H^t(p)}$.

$$\begin{aligned}
\Rightarrow \frac{d}{dt}((\varphi_H^t)^* \omega)_p &= \lim_{s \to 0} \frac{1}{s} [((\varphi_H^{t+s})^* \omega)_p - ((\varphi_H^t)^* \omega)_p] \\
&= \lim_{s \to 0} \frac{1}{s} [(\varphi_H^t)^* ((\varphi_H^s)^* \omega)_{\varphi_H^t(p)} - (\varphi_H^t)^* (\omega_{\varphi_H^t(p)})] \\
&= \lim_{s \to 0} (\varphi_H^t)^* \left(\frac{((\varphi_H^s)^* \omega)_{\varphi_H^t(p)} - \omega_{\varphi_H^t(p)}}{s} \right) \\
&= (\varphi_H^t)^* (\mathcal{L}_{X_H} \omega)_{\varphi_H^t(p)}
\end{aligned}$$

Then prove: $\mathcal{L}_{X_H} \omega = 0$.

$$\Rightarrow \mathcal{L}_{X_H} \omega = d(\iota_{X_H} \omega) + \cancel{\iota_{X_H} d\omega} = d(-dH) = -d^2 H = 0.$$

So $\frac{d}{dt}((\varphi_H^t)^* \omega)_p = 0, \ \forall t \Rightarrow ((\varphi_H^t)^* \omega)_p = ((\varphi_H^0)^* \omega)_p = \omega_p, \ \forall t.$

Since p is arbitrary, then $(\varphi_H^t)^* \omega = \omega$. #.

PROBLEM 6.

Pf (a) WHITNEY'S THEOREM:

For every smooth manifold N^k, it can be embedded into $\mathbb{R}^{2k+1}$.

STRONG WHITNEY'S THEOREM:

Every smooth manifold N^k can be embedded into $\mathbb{R}^{2k}$.

(b) Step 1: Since M^n is compact, it can be covered by finitly many cordinate charts

$$\{(\varphi_i, U_i)\}_{i=1}^{k}.$$

Let $\{\rho_i\}_{i=1}^{k}$ be the P.O.U. belongs to the cover $\{U_i\}_{i=1}^{k}$.

Step 2: Define $f: M \to \mathbb{R}^{nk+k}$ as

$$p \longmapsto ((\rho_1\varphi_1)(p), \cdots, (\rho_k\varphi_k)(p), \rho_1(p), \cdots, \rho_k(p))$$

It's well define since each $\varphi_i(p) \in \mathbb{R}^n$

& if $p \notin U_i$, then $(\rho_i\varphi_i)(p) = 0$.

The smoothness follows from the definition.

<u>Objectivity</u>: If $f(p)=f(q)$, then:

① $\forall i$, $\rho_i(p)=\rho_i(q)$. So: $p\in U_i \Longleftrightarrow q\in U_i$

② For i s.t. $\rho_i(p)\neq 0$, $p\in U_i$, then $q\in U_i$

& $\rho_i(p)\varphi_i(p)=\rho_i(q)\varphi_i(q)$

$\Rightarrow \varphi_i(p)=\varphi_i(q) \Rightarrow p=q$.

<u>f is a immersion</u>: $\forall p$, $\exists i$ s.t. $p\in U_i$ & $\rho_i(p)\neq 0$

<u>Compute</u>: df_p in local chart U_i

$$f\circ\varphi_i(x)=(\cdots,\tilde{\rho}_i(x)x,\cdots,\tilde{\rho}_i(x),\cdots)$$

$\tilde{\rho}_i=\rho_i\circ\varphi_i^{-1}$ cordinate

$$\mathrm{Jac}(f)=\begin{pmatrix} * \\ \tilde{\rho}_i+x^1\frac{\partial\tilde{\rho}_i}{\partial x^1}, & x^1\frac{\partial\tilde{\rho}_i}{\partial x^2}, & \cdots, & x^1\frac{\partial\tilde{\rho}_i}{\partial x^n}, \\ x^2\frac{\partial\tilde{\rho}_i}{\partial x^1}, & \tilde{\rho}_i+x^2\frac{\partial\tilde{\rho}_i}{\partial x^2}, & \cdots, & x^2\frac{\partial\tilde{\rho}_i}{\partial x^n}, \\ \vdots & \vdots & & \\ x^n\frac{\partial\tilde{\rho}_i}{\partial x^1}, & x^n\frac{\partial\tilde{\rho}_i}{\partial x^2}, & \cdots, & \tilde{\rho}_i+x^n\frac{\partial\tilde{\rho}_i}{\partial x^n} \\ * \\ \frac{\partial\tilde{\rho}_i}{\partial x^1}, & \frac{\partial\tilde{\rho}_i}{\partial x^2}, & \cdots, & \frac{\partial\tilde{\rho}_i}{\partial x^n} \\ * \end{pmatrix}$$

So

$\mathrm{Jac}(f,p) \sim$ (equivalent)

$$\begin{pmatrix} * \\ \tilde{\rho}_i I_n \\ * \\ d\tilde{\rho}_i \\ * \end{pmatrix}$$ full rank

$\mathrm{Jac}(f,p)$ is of full rank since at p, $\rho_i(p)\neq 0$

So f is a immersion.

Step3: f is an embedding.

Conclusion: $f: X \to Y$ bijective continuous map,

X, Y are compact, Hausdorff spaces

Then f is homeomorphism.

$\Rightarrow$ So $f: M \to f(M)$ is homeomorphism

subspace topology

$f(M) \subset \mathbb{R}^{nk+k}$

cpt & Hausdorff

Pf of conclusion: Only need to show: f is a closed map, i.e. if $U \overset{\text{closed}}{\subset} X$, then $f(U) \overset{\text{closed}}{\subset} Y$.

X cpt, T_2 & $U \overset{\text{closed}}{\subset} X \Rightarrow U$ cpt $\Rightarrow f(U)$ cpt.

Y cpt, T_2 & $f(U) \overset{\text{cpt}}{\subset} Y \Rightarrow f(U)$ closed.

So: f bijective $\Rightarrow f^{-1}$ exists. #

$A \overset{\text{closed}}{\subset} X$, $(f^{-1})^{-1}(A) = f(A) \overset{\text{closed}}{\subset} X \Rightarrow f^{-1}$ cts

From step $1 \sim 3$, $M \hookrightarrow \mathbb{R}^{nk+k}$. #.

PROBLEM 7.

Sol (a) Recall In HW2, Ex 6, x is decomposable $\overset{\text{iff}}{\iff}$

$$\operatorname{rk}(\text{matrix of coeiffcients of } x) = 1.$$

Only to check: $A = (2024i + 5003j)_{2\times 3}$, $\operatorname{rk}(A) \overset{?}{=} 1$

$$\Rightarrow \quad A = \begin{pmatrix} 2024+5003 & 2024+5003\times 2 & 2024+5003\times 3 \\ 4048+5003 & 4048+5003\times 2 & 4048+5003\times 3 \end{pmatrix}$$

$\operatorname{rk}(A) = 2$, since $\det\begin{pmatrix} 2024+5003 & 2024+5003\times 2 \\ 4048+5003 & 4048+5003\times 2 \end{pmatrix} \neq 0$.

(Or, $\begin{pmatrix} 2024+5003 \\ 4048+5003 \end{pmatrix}, \begin{pmatrix} 2024+5003\times 2 \\ 4048+5003\times 2 \end{pmatrix}$ are linearly independent).

So x is not decomposable.

(b) Let $\{e_1, e_2, e_3\}$ ONB of $\mathbb{R}^3$, $\{e^1, e^2, e^3\}$ related dual basis.

Take $x := e^1 \otimes e^2 \otimes e^2$.

If $\exists\, a \in \Sigma^3(\mathbb{R}^3)^*$, $b \in \Lambda^3(\mathbb{R}^3)^*$ s.t $x = a + b$

Then $\operatorname{Sym}(x) = a$, $\operatorname{Alt}(x) = b$.

$$\Rightarrow \operatorname{Sym}(x) = \frac{1}{6}\sum_{\sigma\in S_3} \sigma \circ (e^1 \otimes e^2 \otimes e^2)$$

$$= \frac{1}{3}(e^1 \otimes e^2 \otimes e^2 + e^2 \otimes e^1 \otimes e^2 + e^2 \otimes e^2 \otimes e^1) = a$$

$$\operatorname{Alt}(x) = \frac{1}{6}\sum_{\sigma\in S_3} (-1)^\sigma \sigma \circ (e^1 \otimes e^2 \otimes e^2) = 0 = b$$

Then $x = a + b = a = \frac{1}{3}(e^1 \otimes e^2 \otimes e^2 + e^2 \otimes e^1 \otimes e^2 + e^2 \otimes e^2 \otimes e^1)$ contradiction!

PROBLEM 8 (Let $\dim M = n$)

Sol (a) STOKE'S THEOREM:

Let M^n be a compact, oriented smooth manifold with boundary ∂M, $i : \partial M \hookrightarrow M$ be the inclusion map, then for any $\omega \in \Omega^{n-1}(M)$,

$$\int_M d\omega = \int_{\partial M} i^*\omega .$$

(b) $\theta = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy \in \Omega^2(\mathbb{R}^3)$.

$d\theta = 3\underbrace{dx\wedge dy\wedge dz}$ $\rightarrow$ standard volume form of $\mathbb{R}^3$.

Let $\mathbb{D}^3 \subset \mathbb{R}^3$ be the closed unit ball, then $\partial \mathbb{D}^3 = S^2$. By Stoke's formula,

$$\int_{S^2} \theta = \int_{\mathbb{D}^3} d\theta = 3\int_{\mathbb{D}^3} dx\wedge dy\wedge dz$$
$$= 3\,\text{volume}(\mathbb{D}^3) = 3\cdot\frac{4}{3}\pi = 4\pi .$$

(c) Prove by contradiction. If $\exists F : M \to \partial M$ smooth & $F|_{\partial M} = \mathrm{Id}_{\partial M}$, then for any $\omega \in \Omega^{n-1}(\partial M)$:

$$\int_{\partial M} \omega = \int_{\partial M} (F|_{\partial M})^*\omega = \int_{\partial M} i^*F^*\omega$$
$$= \int_M d(F^*\omega) = \int_M F^*(d\omega) = 0 \quad \text{since } d\omega = 0$$

contradiction since ∂M is closed, oriented. #

PROBLEM 9

Pf. (a) Basic topology.

M compact & connected + $F \in C^\infty(M)$ (F continuous)

$\Rightarrow F(M) \subset \mathbb{R}$ compact & connected

i.e. $\exists\, m, M \in \mathbb{R}$, s.t. $(M) = [m, M]$

$\Rightarrow \max_M F, \min_M F$ exists, $M = \max_M F$, $m = \max_M F$.

(b) For p: take a local chart (φ, U) near p,

under the local chart, we have:

$$F \circ \varphi^{-1} \leq \max_M F = F(p), \quad \text{from } U \subset \mathbb{R}^m \text{ to } \mathbb{R}$$

so $\varphi(p)$ is the maximum of $F \circ \varphi^{-1}$, $\varphi(p)$

is a critical pt of $F \circ \varphi^{-1}$, i.e.

$$d(F \circ \varphi^{-1})|_{\varphi(p)} = 0, \rightarrow \text{Fermat's Lem}$$

that is, $dF_p \circ (d\varphi_p)^{-1} = 0$. So $dF_p = 0$.

Simiarly, q is a critical pt of F.

(c) If $p \neq q$, then done;

If $p = q$, then $F \equiv$ constant $\Rightarrow$ every pt is critical pt. #

PROBLEM 10.

Sol. (a) $\mathbb{C}P^n = (\mathbb{C}^{n+1}\setminus\{0\})/\sim$, where $z, w \in \mathbb{C}^{n+1}\setminus\{0\}$, $z \sim w \iff \exists \lambda \in \mathbb{C}^*$, $z = \lambda w$.

Define: $\overline{A} : \mathbb{C}P^n \to \mathbb{C}P^n$, $[z] \mapsto [Az]$

Check $\overline{A}$ is well defined.

If $[z_1] = [z_2]$, then $\exists \lambda \in \mathbb{C}^*$, $z_1 = \lambda z_2$.

$[Az_1] = [\lambda A z_2] = [Az_2]$

(Note $A \in GL(n+1, \mathbb{C})$, for $z \in \mathbb{C}^{n+1}\setminus\{0\}$, $Az \neq 0$)

Recall the charts on $\mathbb{C}P^n$: $\{(\varphi_i, U_i)\}_{i=1}^{n+1}$, where:

$U_i := \{[z_1, \cdots, z_{n+1}] \in \mathbb{C}P^n : z_i \neq 0\}$,

$\varphi_i^{-1} : (z_1, \cdots, z_n) \mapsto [z_1, \cdots, z_i, 1, z_{i+1}, \cdots, z_n]$.

On U_i & U_j, check: $\qquad w^i = (z_1, \cdots, z_i, 1, z_{i+1}, \cdots, z_n)$

$$\varphi_j \circ \overline{A} \circ \varphi_i^{-1} : (z_1, \cdots, z_n) \mapsto \left(\frac{(Aw^i)_1}{(Aw^i)_j}, \cdots, \frac{(Aw^i)_{j-1}}{(Aw^i)_j}, \frac{(Aw^i)_{j+1}}{(Aw^i)_j}, \cdots, \frac{(Aw^i)_{n+1}}{(Aw^i)_j} \right)$$

is a smooth map. So is $\overline{A}$. (Actually is holomorphic).

#

(b) Fixed pts of $\overline{A}$: $[Az] = [z]$

i.e. $\exists \lambda \in \mathbb{C}^*$, $Az = \lambda z \iff z$ is a eigenvector of A.

#

(c) Since $A \in GL(n+1, \mathbb{C})$, $Az \sim Aw$ iff $z \sim w$

Write $A = P\,\text{diag}\{\lambda_1, \cdots, \lambda_{n+1}\}P^{-1}$, using linear transform on $\mathbb{C}^{n+1}$, we may assume $A = \text{diag}\{\lambda_1, \cdots, \lambda_{n+1}\}$.

For $p_i = [0, \cdots, 0, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, 0, \cdots, 0]$, p_i is a fix point of $\bar{A}$.

$\text{Fix}(\bar{A}) = \{p_i\}_{i=1}^{n+1}$

$$A(0, \cdots, 0, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, 0, \cdots, 0)^T = \lambda_i (0, \cdots, 0, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, 0, \cdots, 0)^T$$

Using chart near p_i, then:

$$\bar{A} : (z_1, \cdots, z_n) \longmapsto \left(\frac{\lambda_1 z_1}{\lambda_i}, \cdots, \frac{\lambda_{i-1} z_{i-1}}{\lambda_i}, \frac{\lambda_{i+1} z_i}{\lambda_i}, \cdots, \frac{\lambda_{n+1} z_n}{\lambda_i} \right)$$

And the complex Jacobian of $\bar{A}$ near p is:

$$\text{Jac}_{\mathbb{C}}(\bar{A}) = \text{diag}\left\{ \frac{\lambda_1}{\lambda_i}, \cdots, \frac{\lambda_{i-1}}{\lambda_i}, \frac{\lambda_{i+1}}{\lambda_i}, \cdots, \frac{\lambda_{n+1}}{\lambda_i} \right\}$$

Rmk Complex Jacobian for $\varphi : \mathbb{C}^k \to \mathbb{C}^k$, $\text{Jac}_{\mathbb{C}}(\varphi) = \left(\frac{\partial \varphi_i}{\partial z_j} \right)_{n \times n}$

φ is holomorphic, if $\frac{\partial \varphi_i}{\partial \bar{z}_j} = 0$, $\forall i, j$.

For holomorphic map φ, see $\varphi : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$, its real Jacobian is

$$\text{Jac}_r(\varphi) = \begin{pmatrix} \text{Re}\,\text{Jac}_{\mathbb{C}}(\varphi) & -\text{Im}\,\text{Jac}_{\mathbb{C}}(\varphi) \\ \text{Im}\,\text{Jac}_{\mathbb{C}}(\varphi) & \text{Re}\,\text{Jac}_{\mathbb{C}}(\varphi) \end{pmatrix}$$

Lem $\det\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = |\underbrace{\det(A + iB)}_{\text{complex matrix}}|^2$.

So at p_i, the following holds

$$\mathrm{Jac}_{\mathbb{C}}(\bar{A}, p_i) - Id = \mathrm{diag}\left\{\frac{\lambda_1-\lambda_i}{\lambda_i}, \cdots, \frac{\lambda_{i-1}-\lambda_i}{\lambda_i}, \frac{\lambda_{i+1}-\lambda_i}{\lambda_i}, \cdots, \frac{\lambda_{n+1}-\lambda_i}{\lambda_i}\right\}.$$

$$\Rightarrow \det(\mathrm{Jac}_{\mathbb{R}}(\bar{A}, p_i) - Id)$$

$$= |\det(\mathrm{Jac}_{\mathbb{C}}(\bar{A}, p_i) - Id)|^2$$

$$= \left|\prod_{j\neq i} \frac{\lambda_j-\lambda_i}{\lambda_i}\right|^2 = \prod_{j\neq i} \frac{|\lambda_j-\lambda_i|^2}{|\lambda_i|^2} > 0$$

so $\bar{A}$ is Lefschetz map.

eigenvalues of A all have multiplicity $1 \Rightarrow \lambda_i \neq \lambda_j$ if $i \neq j$.

#

(d) In (c), we can see that $\det(d\bar{A}_{p_i} - Id) > 0$, so:

$$L(\bar{A}) = \sum_{i=1}^{n+1} \mathrm{sign}(\det(d\bar{A}_{p_i} - Id))$$

$$= \sum_{i=1}^{n+1} 1 = n+1.$$

<u>Rmk</u> $\chi(\mathbb{C}P^n) = n+1$.